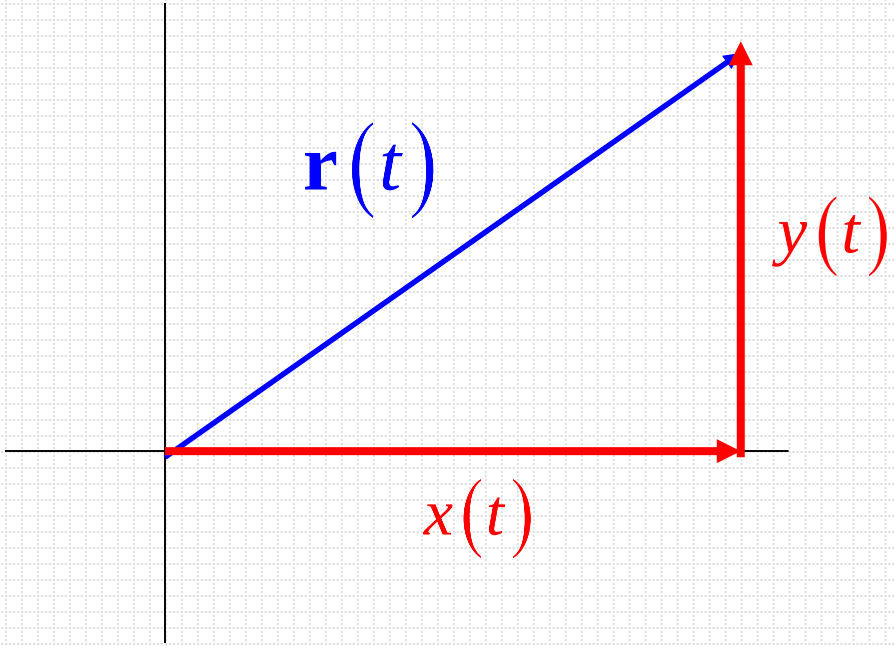
Section 10.3

Vector Valued Functions

We can describe the position of a moving particle by a *vector* in component form. Vectors have a direction and magnitude.



$$r(t) = \langle x(t), y(t) \rangle$$

The position is composed of horizontal and vertical **components**.

The length of the vector is called the **magnitude** and is denoted by $|r(t)|$

$$|r(t)| = \sqrt{(x(t))^2 + (y(t))^2}$$

PVA Problems in the Vector/Parametric World

Position Vector: $(x(t), y(t))$ or $\langle x(t), y(t) \rangle$

Velocity Vector: $v(t) = \langle x'(t), y'(t) \rangle$

Slope of tangent line: $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

Object at rest: $x'(t) = 0$ AND $y'(t) = 0$ SIMULTANEOUSLY

Speed: $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

PVA Problems in the Vector/Parametric World

Acceleration Vector: $a(t) = \langle x''(t), y''(t) \rangle$

Speeding up:
x' AND x'' HAVE SAME SIGN → SPEEDING UP HORIZONTALLY
y' AND y'' HAVE SAME SIGN → SPEEDING UP VERTICALLY.

Slowing down:
x' AND x'' OPPOSITE SIGN → SLOWING DOWN HORIZONTALLY
y' AND y'' OPPOSITE SIGN → " " VERTICALLY

Other key formulas:

SMELLS LIKE ARC LENGTH!

Total Distance Traveled: $\int |v(t)|\,dt = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

Final Position = Initial Position + Displacement

$x(t_2) = x(t_1) + \int_{t_1}^{t_2} x'(t)\,dt$

$y(t_2) = y(t_1) + \int_{t_1}^{t_2} y'(t)\,dt$

Example 5: Let the position of an object be given by the vector to the right:

$$\langle \underbrace{3\cos t}_{x(t)}, \underbrace{3\sin t}_{y(t)} \rangle$$

a) Find the velocity and acceleration vectors.

$v(t) = \langle x'(t), y'(t) \rangle = \langle -3\sin t, 3\cos t \rangle$

$a(t) = \langle x''(t), y''(t) \rangle = \langle -3\cos t, -3\sin t \rangle$

b) Find the velocity, acceleration, speed and direction of motion at $t = \frac{\pi}{4}$.

$v(\pi/4) = \langle \frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle$

$a(\pi/4) = \langle \frac{-3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2} \rangle$

SPEED $= \sqrt{\left(\frac{-3\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2} = 3$

DIRECTION: $x'(\pi/4) < 0 \Rightarrow$ LEFT $\quad y'(\pi/4) > 0 \Rightarrow$ UP

SPEEDING UP IN X-DIRECTION: $x'(\pi/4)$ AND $x''(\pi/4)$ BOTH NEG.

SLOWING DOWN IN Y-DIRECTION: $y'(\pi/4)$ AND $y''(\pi/4)$ ARE OPP. SIGNS.

Example 6: Given the position vector: $\langle 2t^3 - 3t^2, t^3 - 12t \rangle$

$x(t)$ $y(t)$

a) Write the equation of the tangent when $t = -1$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-12}{6t^2-6t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = \frac{-9}{12} = \frac{-3}{4} \Rightarrow \text{SLOPE}$$

$$\text{POINT}: \left(x(-1), y(-1) \right) = (-5, 11)$$

$$\boxed{y - 11 = -\frac{3}{4}(x+5)}$$

Example 6: Given the position vector: $\langle 2t^3 - 3t^2, t^3 - 12t \rangle$

b) Find the coordinates of all points where the horizontal component of the velocity is zero.

$x'(t) = 0$

$6t^2 - 6t = 0$

$6t(t-1) = 0$

$t = 0, 1$

$(x(0), y(0)) = \boxed{(0,0)}$

$(x(1), y(1)) = \boxed{(-1, -11)}$

Example 6: Given the position vector: $\langle 2t^3 - 3t^2, t^3 - 12t \rangle$

$x' = 6t^2 - 6t \quad y' = 3t^2 - 12$

c) Find the total distance traveled on the interval [0,2]. (CALCULATOR)

$$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \approx 19.2599\ldots$$

$$\int_0^2 \sqrt{(6t^2 - 6t)^2 + (3t^2 - 12)^2}\, dt$$

Example 6: Given the position vector: $\left\langle 2t^3 - 3t^2, t^3 - 12t \right\rangle$

d) For what value(s) of t is the object at rest, if any?

$x' = 0$ AND $y' = 0$ SIMULTANEOUSLY

$6t^2 - 6t = 0$ $\quad$ $3t^2 - 12 = 0$

$6t(t-1) = 0$ $\quad$ $3(t^2-4) = 0$

$t = 0, 1$ $\quad$ $t = \pm 2$

NEVER AT REST.

Example 9: $v(t) = \left(\frac{1}{t+1}, 2t\right)$

$x' \quad y'$

a) Find the acceleration vector and speed at $t = 2$.

$$a = \langle x''(t), y''(t) \rangle = \left\langle \frac{-1}{(t+1)^2}, 2 \right\rangle \Rightarrow a(2) = \left\langle \frac{-1}{9}, 2 \right\rangle$$

$$\text{SPEED} = \sqrt{(x'(2))^2 + (y'(2))^2} = \sqrt{\left(\frac{1}{3}\right)^2 + (4)^2} = \sqrt{\frac{1}{9} + 16} = \sqrt{\frac{145}{9}}$$

b) (Calculator) Find the total distance traveled from $t = 0$ to $t = 2$.

$$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 4.335$$

Example 9: $v(t) = \left(\frac{1}{t+1}, 2t \right)$

$a(t) = \left\langle \frac{-1}{(t+1)^2}, 2 \right\rangle$

$x'' < 0$

c) If $x(3) = -4$, find the x – coordinate of the object at $t = 6$.

$$x(6) = x(3) + \int_3^6 x'(t)\,dt$$

$$= -4 + \int_3^6 \frac{1}{t+1}\,dt = -4 + \ln|t+1| \Big]_3^6 = \boxed{-4 + \ln 7 - \ln 4}$$

d) Describe the motion of the object as *t* increases without bound.

$\lim_{t \to \infty} x'(t) = 0$ — $x' > 0$ AND APPROACHING 0 AS $t \to \infty$ ⇒ MOVING RT, SLOWING DOWN.

$\lim_{t \to \infty} y'(t) = \infty$ — $y' > 0$ $(t \geq 0)$ AND APPROACHING ∞ AS $t \to \infty$ ⇒ OBJECT MOVING UP, SPEEDING UP.

$y'' > 0$ $(t \geq 0)$

Classwork:

AP Packet #30, 32

Homework:

AP Packet #23 - 31